# New Concepts for Finite-Element Mass Matrix Formulations

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**Improved finite-element formulations are presented for dynamic analysis. The improvements are based on modifications of the finite-element mass-matrix and are of high importance, since no additional computational effort is required, compared with the necessary effort for the standard, consistent mass-matrix formulation. Formulations are presented for the one-dimensional bar, two-dimensional membrane, and the pure bending beam element. Free vibration analysis and forced vibration results are presented employing new mass matrices. In general, the frequency errors and stress errors are at least three times smaller than the corresponding consistent or lumped-mass formulation errors. Accurate model data can be achieved for all the modes of relatively coarse meshes.**

## Introduction

IN finite-element dynamic analysis of structures, low-order and high-order frequency expansion approaches can be employed for the formulation of finite-element mass and stiffness matrics, Ref. 7.

In general, the low-order approach is extensively employed in finite-element programs, and the mass discretization results in the lump-mass formulation and the consistent mass formulation for the eigenvalue problem

$$(K-\omega_n^2 M)Q_n = 0 \tag{1}$$

In the higher-order approach, it is assumed that the finite-element shape functions depend on the unknown frequency $\omega$. In Refs. 3 and 4 Gupta obtains mass and stiffness matrices that are series expansion in

$$[K] = [K_0+\omega^4 K_4 + \ldots] \tag{2}$$

$$[M] = [M_0+\omega^2 M_2 + \ldots] \tag{3}$$

which result in the quadratic eigenproblem

$$[K_0-\omega^2 M_0-\omega^4(M_2-K_4)]Q = \{0\} \tag{4}$$

Although the algebraic manipulations and the eigenvalue extraction appear to be more cumbersome compared with the lower-order approach, the problem solution time for the two cases is about the same.[4] In general, though, this method was not implemented in commercially available finite-element programs, until recently.[6]

It is considered appropriate to improve current lower-order finite-element discretization techniques for the dynamic response of a structural system in order to provide 1) a better modal representation, i.e., frequencies and mode shapes, and 2) a better dynamic response, i.e., displacement and stress resulting from dynamic loads.

The improvements presented are based on modifications of the element mass matrix and require no additional computational effort, compared with the standard mass formulation.

## Problem Description

In considering the eigenvalue problem for a selected set of dynamic degrees of freedom, the lower-order approach also results in the frequency-dependent eigenvalue problem

$$\left[-\omega^2\begin{bmatrix} M_{bb} & M_{bi} \\ M_{ib} & M_{ii}\end{bmatrix} + \begin{bmatrix} K_{bb} & K_{bi} \\ K_{ib} & K_{ii}\end{bmatrix}\right]\begin{Bmatrix} u_b \\ u_i\end{Bmatrix} = \begin{Bmatrix} 0 \end{Bmatrix} \tag{5}$$

where $\{u_b\}$ are the dynamic degrees of freedom and $\{u_i\}$ the dependent degrees of freedom for the dynamic problem. Solving for $\{u_i\}$,

$$\{u_i\} = -(-\omega^2 M_{ii}+K_{ii})^{-1}(-\omega^2 M_{ib}+K_{ib})\{u_b\} \tag{6}$$

the resulting eigenvalue problem for the dynamic degrees of freedom

$$\begin{aligned}&[-\omega^2 M_{bb}+K_{bb}-(-\omega^2 M_{bi}+K_{bi})(-\omega^2 M_{ii}+K_{ii})^{-1}\\ &\times(-\omega^2 M_{ib}+K_{ib})]\{u_b\} = \{0\}\end{aligned} \tag{7}$$

implies an eigenvalue problem of the form

$$[A-\omega^2 B-\omega^4 C\ldots]\{u_b\} = \{\emptyset\} \tag{8}$$

where $A$, $B$, $C$ are functions of the original partitioned matrices in Eq. (5).

Since a discretized dynamic problem is a reduced set of the continuous structural formulation, the form of the eigenvalue problem in Eq. (1) is an assumption for the general continuous problem in Eq. (8). From engineering considerations, it is important to employ the static stiffness matrix in the eigenvalue equation (1). Therefore, there is a need to investigate the errors arising from the mass matrix, especially for cases where the static stiffness is accurately, if not exactly, representing the continuous problem. Such a case arises for the one-dimensional bar element, where the static discretization corresponds to the elastic continuum. The consistent and lumped-mass eigenvalue problem is presented in Fig. 1.

It appears that the frequencies for the consistent mass approach correspond to an upper bound, and the frequencies for the lumped-mass approach correspond to a lower bound. Because the static stiffness is exact, it is considered approprite to

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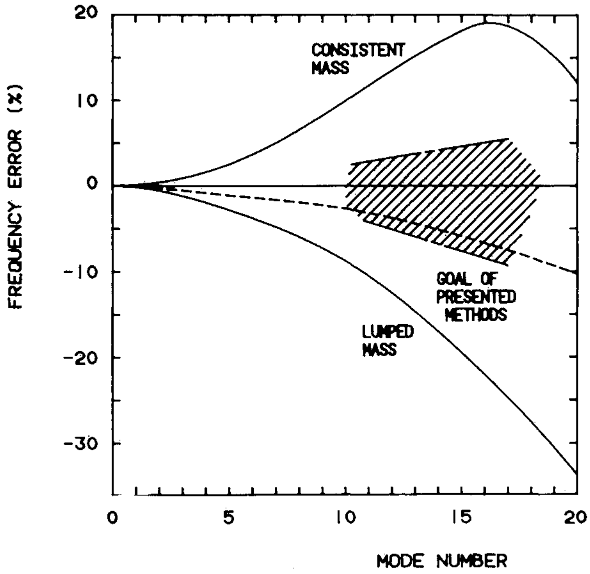


**Fig. 1 Frequency error vs mode number in classical approaches for a 20-bar element model and goal of the presented methods.**

reduce the error in modal data by considering an intermediate mass matrix formulation. In Ref. 1 the spring mass matrix

$$[M] = \rho \frac{AL}{8} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \tag{9}$$

is presented by considering the element shape functions $\varphi_1 = \cos^2 T\Theta$ and $\varphi_2 = \sin^2\Theta$.

Accurate modal and response results are achieved for all the modes of relatively course meshes.

## One-Dimensional Bar Element

In the case of equal length elements, the error analysis is considered in two ways: 1) using a substitution method based on Taylor series and 2) using the properties of Toeplitz matrices.

The element mass-matrix formulation can be expressed as

$$[M(a)] = \rho AL \begin{bmatrix} (0.5-a) & a \\ a & (0.5-a) \end{bmatrix} \tag{10}$$

The formulation for the lumped mass corresponds to $a = 0$, the consistent mass to $a = 1/6$ and nonconsistent mass to all other values as $a = 1/8$ (Stavrinidis' method) and $1 = 1/12$ (average of lumped and consistent). In the case of equal length elements, the rows of the matrix in Eq. (1) are all similar except for the first and last due to the geometry edge conditions. The $j$th equation is

$$\ddot{a}u_{j-1} + (1-2a)\ddot{u}_j + a\ddot{u}_{j+1} = \frac{c^2(u_{j+1} - 2u_j + u_{j-1})}{\Delta x^2} \tag{11}$$

The length $L$ is replaced by $\Delta x$ for the Taylor series expansions. Assuming $u_j$ in Eq. (11) is the correct analytical solution at mode $j$, then $u_{j-1}$ and $u_{j+1}$ is formulated around $x_j$ and substituted in Eq. (11) as in Ref. 2:

$$\ddot{u} + a\frac{\delta^2\ddot{u}}{\delta x^2}\Delta x^2 + \frac{1}{12}a\frac{\delta^4\ddot{u}}{\delta x^2}\Delta x^4 \ldots$$

$$= \frac{c^2}{\Delta x^2}\left(\frac{\delta^2 u}{\delta x^2}\Delta x^2 + \frac{1}{12}\frac{\delta^4 u}{\delta x^4}\Delta x^4 + \ldots\right) \tag{12}$$

where the index $j$ is omitted for simplicity. Equation (12) represents the wave equation plus higher-order terms, which vanish when $\Delta x$ tends to zero. The mentioned procedure does not provide information for the finite-element solution but tests the accuracy of the discretized equations.

Considering the error of order $\Delta x^2$,

$$\epsilon_1 = \left(\frac{c^2}{12}\frac{\partial u^4}{\partial x^4} - a\frac{\partial^2 \mu}{\partial x^2}\right)\Delta^2 x \tag{13}$$

and employing the wave equation in the form

$$\frac{\partial^2 \ddot{u}}{\partial x^2} = c^2\frac{\partial^4 u}{\partial x^4} \tag{14}$$

where $c^2 = E/\rho$, the error of order $\Delta x^2$ is

$$\epsilon_1 = c^2\frac{\partial^4 u}{\partial x^4}\left(\frac{1}{12} - a\right)\Delta x^2 \tag{15}$$

Taking $a = 1/12$, the error is reduced to order $\Delta x^4$, which results in the improved accuracy for the second-order formulation in Figs. 2 and 3.

In the method using the properties of the Toeplitz matrices, the error is computed as the difference between the analytical solution and an analytical expression for the numerical solution. The latter is obtained by using the Toeplitz structure of the mass and stiffness matrices, where it is essential that the value of each matrix element with indices $ij$ depends only on the difference $i-j$. In Ref. 2 the $n$th frequency $\omega_n$ for the fixed-free case

$$[K] = \frac{EA}{\Delta x}\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & — & — & — & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix} \tag{16a}$$

$$[M] = \rho A\,\Delta x\begin{bmatrix} (1-2a) & a & & & \\ a & (1-2a) & a & & \\ & — & — & — & \\ & & a & (1-2a) & a \\ & & & a & (0.5-a) \end{bmatrix} \tag{16b}$$

is analytically derived

$$\omega_n^2 = \left[(n-½)\pi\frac{C}{L}\right] + \left(\frac{C}{\Delta x}\right)^2\left[(n-½)\pi\frac{\Delta x}{L}\right]^4(a-½)$$

$$+\left(\frac{C}{\Delta x}\right)^2\left[(n-½)\pi\frac{\Delta x}{L}\right]^6\left(a^2 - \frac{a}{6} + \frac{1}{360}\right) + \ldots \tag{17}$$

The exact frequencies $\overline{\omega}_n$ for the continuous problem are

$$\overline{\omega}_n^2 = [(n-½)\pi C/L]^2 \tag{18}$$

As derived previously, when $a = 1/12$, the accuracy for the frequencies is of second order. For simplicity, assigning $\alpha$,

$$(n-½)(\pi\Delta x/L) = \alpha \tag{19}$$

the ratio of the $n$th numerical and analytical eigenvalue

$$\frac{\omega_n^2}{\overline{\omega}_n^2} = \frac{2(1-\cos\alpha)}{\alpha^2(1-2a+2a\cos\alpha)} \tag{20}$$

The minimization of the maximum error for all the frequencies is derived in Ref. 2, where it is shown that the value of

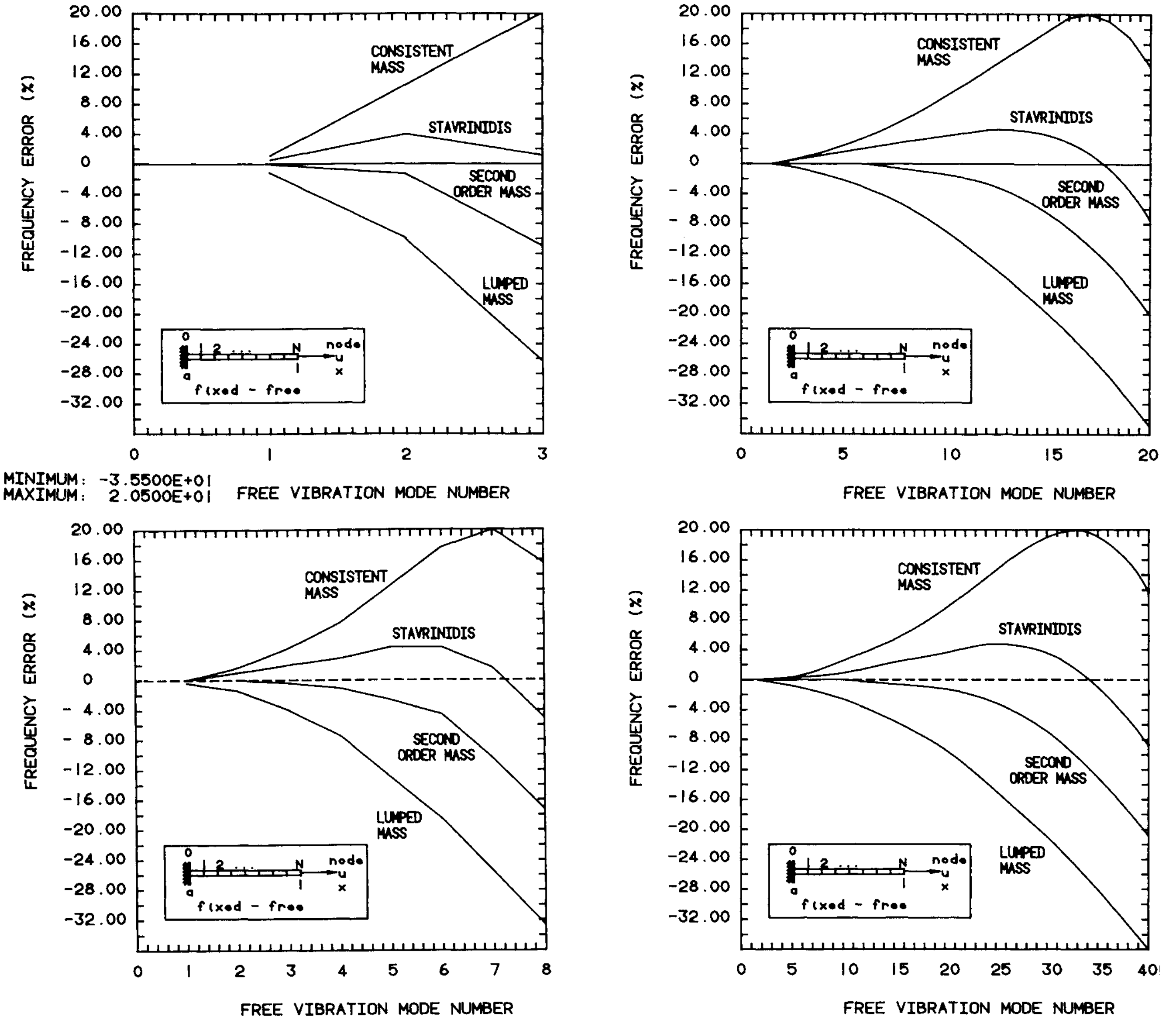


**Fig. 2 Frequency error vs mode number for a 3-, 8-, 20-, and 40-bar element fixed-free model.**

about $a = 1/8$, as proposed by Stavrinidis in Ref. 1, minimizes the maximum errors and that the $a$ value, which minimizes the error, slightly depends on the number of degrees of freedom.

In Fig. 2, the frequency errors are presented for fixed-free vibration analysis considering 3, 8, 20, and 40 elements. In Fig. 3 the corresponding results are presented for the free-free vibration analysis. In Fig. 4 the stress for the forced sinusoidal excitation is presented for eight elements. It can be concluded that significant error reduction can be achieved with the new approaches for modal data and dynamic responses.

## Pure Bending Element

The equation of motion for transverse vibrations $u(x,t)$ of a one-dimensional beam with length $L$, excluding axial and shear deformations and neglecting rotational inertia, is given by

$$EI\frac{\partial^4 u}{\partial x^4} + \rho A\frac{\partial^2 u}{\partial t^2} = 0 \tag{21}$$

for $0 \le x \le L$ and $t \ge 0$, where $I$ is the moment of inertia of the cross section about the neutral axis and $A$ its area. The symbols $E$ and $\rho$ are the elastic modulus and mass per unit volume of the material, respectively. The mass per unit length $\rho A$ and the flexural rigidity $EI$ are assumed constant along the beam.

As in the case of the one-dimensional element, two methods are considered for the error analysis: the substitution method, based on Taylor series, and the properties of Toeplitz matrices. However, the latter is no longer readily applicable because the present element has two degrees of freedom per node, and the element matrices only have a "block-Toeplitz" structure. That is, neglecting boundaries, the element matrices can be partitioned in the form

$$\begin{bmatrix} - & - & - & & \\ & B^T & A & B & \\ & & B^T & A & B \\ & & & - & - \end{bmatrix} \tag{22}$$

where $A$ and $B$ are $2\times 2$ matrices to be determined by assemblage of the element matrices. Both the mass matrix and the stiffness matrix have this structure. The respective cell, in the case of equally spaced nodes and away from the boundaries, consists of the matrix

$$[B^T \quad A \quad B] \tag{23}$$

which is repeated along the diagonal $A$ of the global matrix.

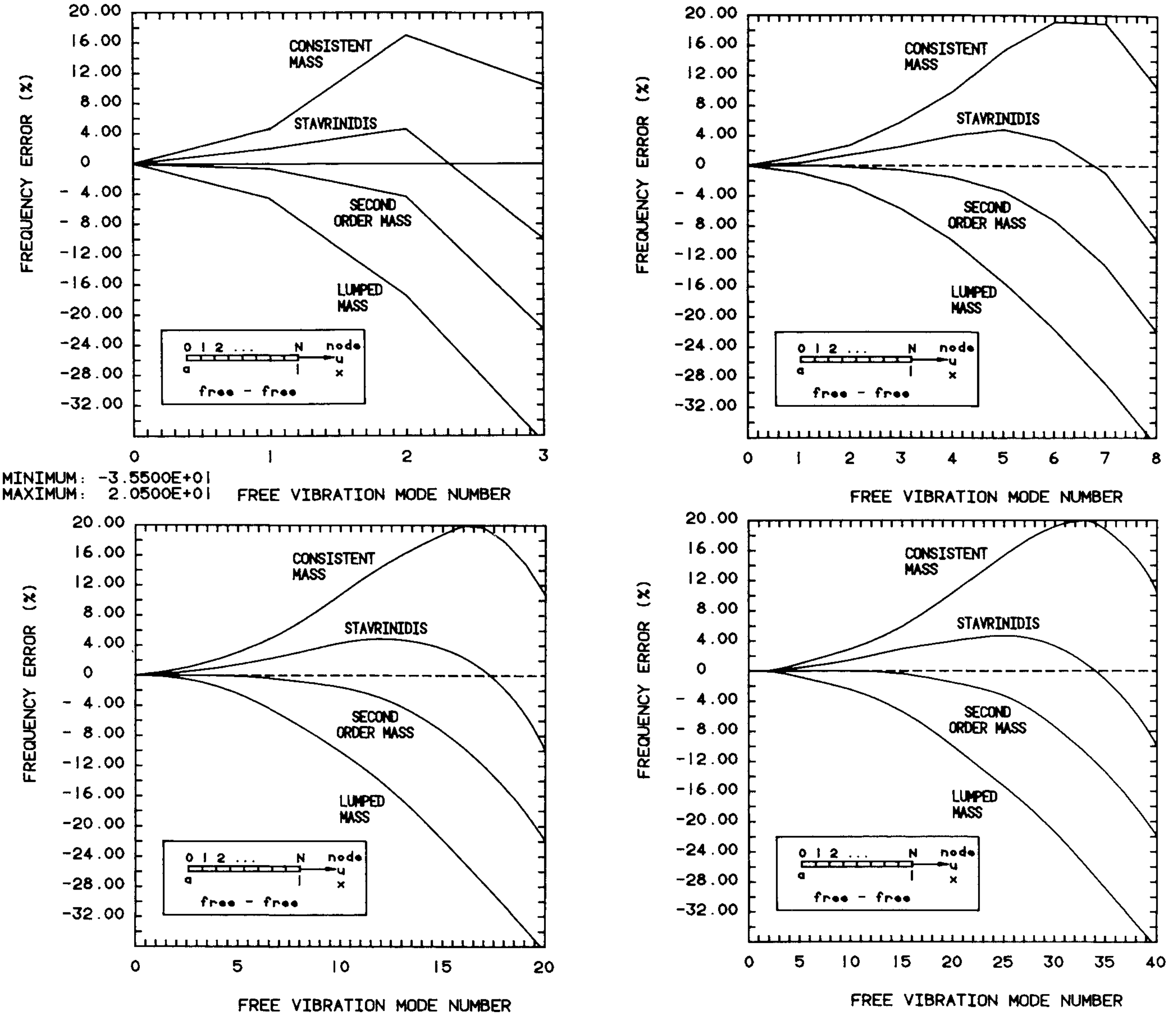


**Fig. 3 Frequency error vs mode number for a 3-, 8-, 20-, and 40-bar element free-free model.**

The repetitive cell for the stiffness matrix

$$\frac{2EI}{\Delta x^3}\begin{bmatrix} -6 & -3\Delta x & 12 & 0 & -6 & 3\Delta x \\ 3\Delta x & \Delta x^2 & 0 & 4\Delta x^2 & -3\Delta x & \Delta x^2 \end{bmatrix} \tag{24}$$

and the repetitive cell for the consistent mass matrix

$$\rho\frac{A\,\Delta x}{420}\begin{bmatrix} 54 & 13\Delta x & 312 & 0 & 54 & -13\Delta x \\ -13\Delta x & -3\Delta x^2 & 0 & 8\Delta x^2 & 13\Delta x & -3\Delta x^2 \end{bmatrix} \tag{25}$$

The odd columns correspond to the translational degrees of freedom $u_{j-1}$, $u_j$, and $u_{j+1}$, whereas the even columns represent the rotational degrees of freedom $\Theta_{j-1}$, $\Theta_j$, and $\Theta_{j+1}$. Substitution of the series as in Eqs. (11) and (12) results in one expression per row in terms of both $\Theta_j$ and $u_j$. It is shown in Ref. 2 that the truncation error for the consistent mass matrix

$$[M] = \rho\frac{A\,\Delta x}{420}\begin{bmatrix} 156 & 22\Delta x & 54 & -13\Delta x \\ & 4\Delta x^2 & 13\Delta x & -3\Delta x^2 \\ \text{(symmetric)} & & 156 & -22\Delta x \\ & & & 4\Delta x^2 \end{bmatrix} \tag{26}$$

is for the series expansion of the first row of the matrices in Eqs. (24) and (25)

$$\frac{1}{720}EI\frac{\partial^8 u}{\partial x^8}\Delta x^5 + 0(\Delta x^7) \tag{27}$$

and for the second row

$$\frac{11}{75{,}600}EI\frac{\partial^9 u}{\partial x^9}\Delta x^7 + 0(\Delta x^9) \tag{28}$$

It is therefore concluded that the consistent mass formulation yields fourth-order accuracy. Error analysis is considered for a general matrix

$$[M] = \rho A\,\Delta x\begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ & M_{22} & M_{23} & M_{24} \\ & & M_{33} & M_{34} \\ \text{(symmetric)} & & & M_{44} \end{bmatrix} \tag{29}$$

where the elements are allowed to vary. The 10 unknowns will be eliminated by imposing symmetry and conservation conditions, accuracy conditions, and limiting the truncation error of the finite-element model for improved modal data.

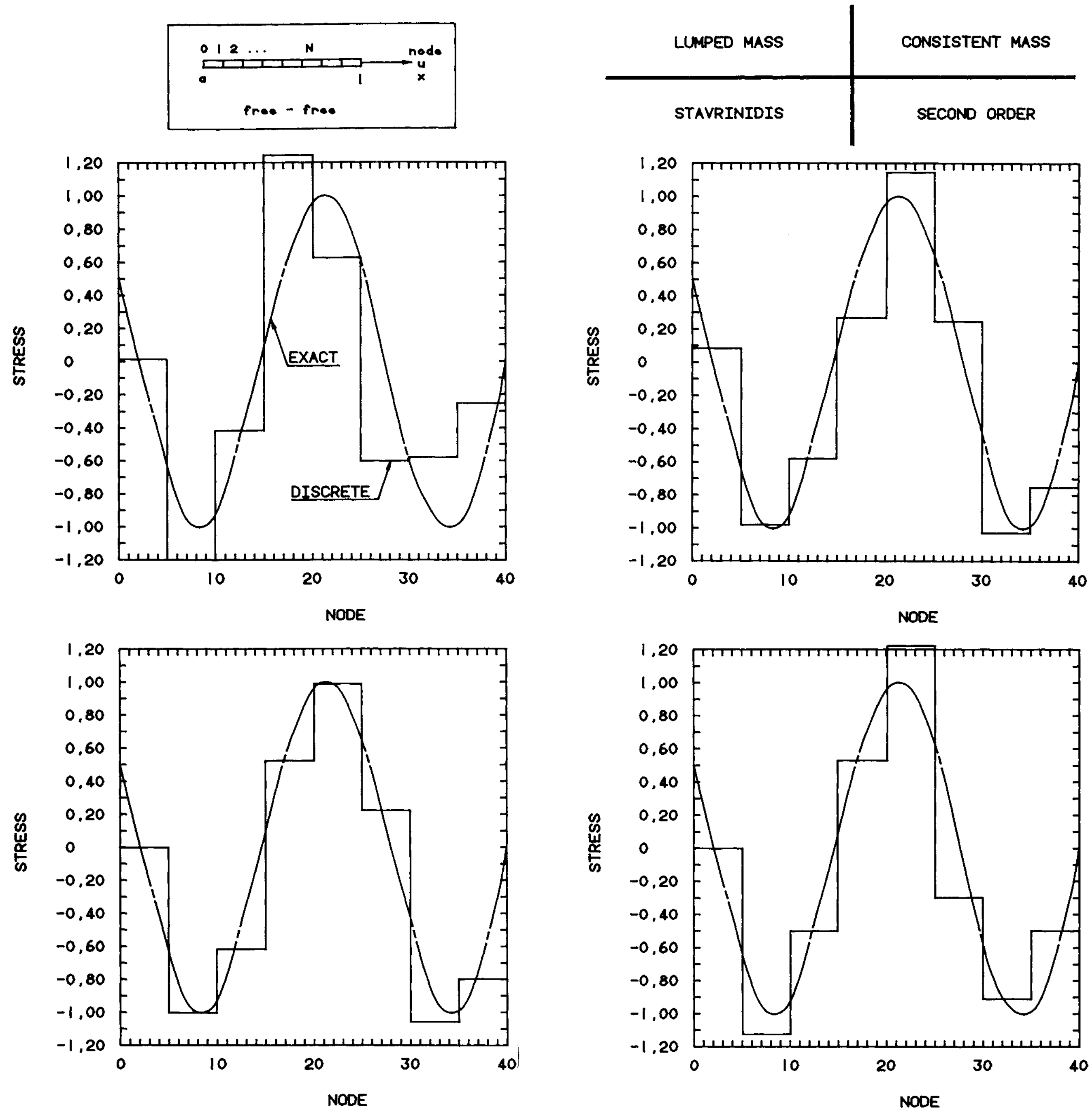


**Fig. 4 Comparison of analytical and finite-element stress for forced sinusoidal excitation of a free-free 8-bar element model.**

It is shown in Ref. 2 that using the Taylor series substitution method as for the one-dimensional bar element results in

$$[M] = \rho\frac{A\,\Delta x}{420}\begin{bmatrix} 163 & 25.5\Delta x & 47 & -9.5\Delta x \\ & 7.5\Delta x^2 & 9.5\Delta x & -3\Delta x^2 \\ (\text{symmetric}) & & 163 & -25.5\Delta x \\ & & & 7.5\Delta x^2 \end{bmatrix} \tag{30}$$

The truncation error for the mass matrix in Eq. (30), compared with the consistent mass matrix truncation error in Eq. (27) and (28),

$$\frac{1}{10{,}800}EI\frac{\partial^{10}u}{\partial x^{10}}\Delta x^7 + 0(\Delta x^9) \tag{31}$$

$$\frac{1}{18{,}900}EI\frac{\partial^9 u}{\partial x^9}\Delta x^7 + 0(\Delta x^9) \tag{32}$$

It is therefore concluded that the new formulation yields sixth-order accuracy, and improvement of two orders of magnitude over current methodologies.

In Fig. 5 the frequency error is presented for the consistent mass (fourth-order accuracy) and for the new mass matrix (sixth-order accuracy). A general accuracy improvement is shown, which might be improved further. Such investigations are in progress, and a mass matrix may be found yielding lower-order accuracy but a smaller maximum frequency error, as was the case with Stavrinidis' formulation for the bar element. The following shape functions might be employed for the element mass matrix:

$$\Phi_1 = \cos^2\Theta$$

$$\Phi_2 = \sin\cos^2\Theta$$

$$\Phi_3 = \sin^2\Theta$$

$$\Phi_4 = \cos\Theta\sin^2\Theta \tag{33}$$

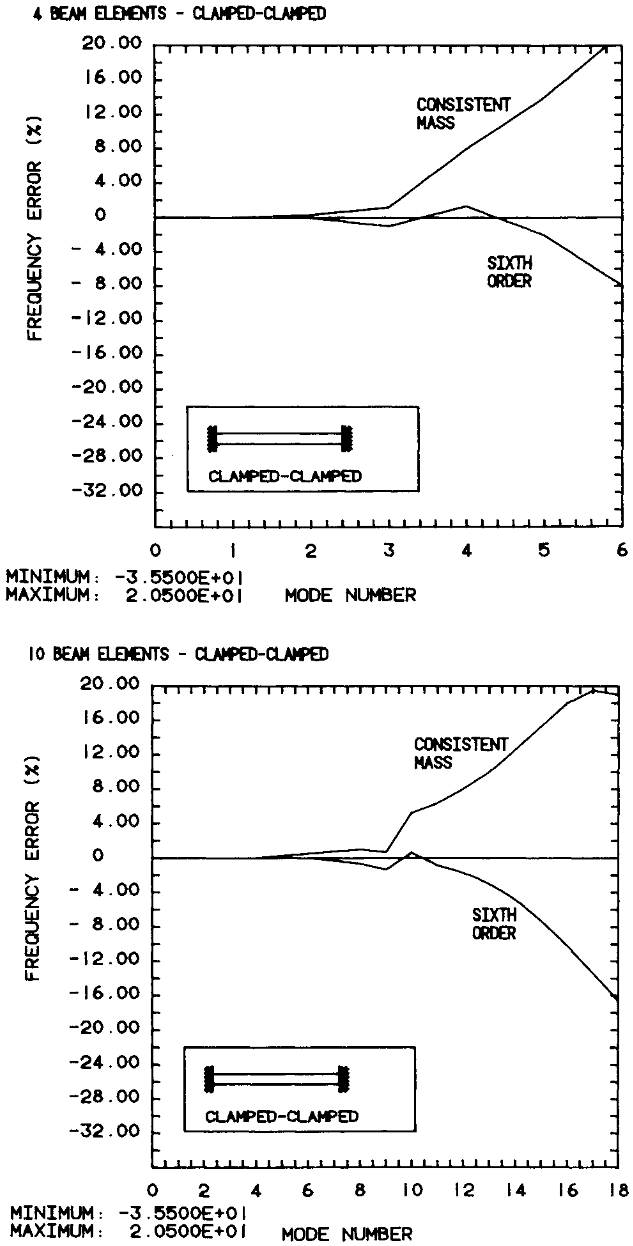


**Fig. 5 Frequency error vs mode number for a 4- and 10-beam element clamped-clamped model.**

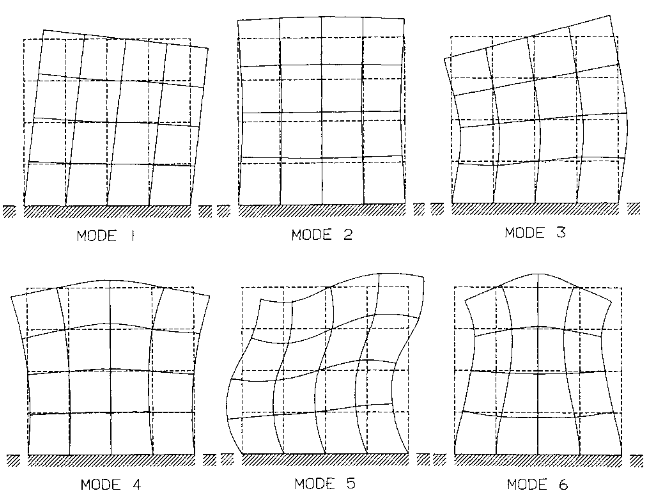


**Fig. 6 Vibration analysis example for a two-dimensional membrane fixed-free model.**

or, in general,

$$\Phi = f(\Theta, \cos\Theta, \sin\Theta, \cosh\Theta, \sinh\Theta \tag{34}$$

## Two-Dimensional Membrane Element

The equations of motion of a two-dimensional elastic continuum freely vibrating in its own plane are

$$(\lambda + 2\mu)\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial^2 u}{\partial y^2} + (\lambda + \mu)\frac{\partial^2 v}{\partial x \partial y} = \rho\frac{\partial^2 u}{\partial t^2} \tag{35a}$$

$$(\lambda + 2\mu)\frac{\partial^2 v}{\partial y^2} + \mu\frac{\partial^2 v}{\partial x^2} + (\lambda + \mu)\frac{\partial^2 u}{\partial x \partial y} = \rho\frac{\partial^2 v}{\partial t^2} \tag{35b}$$

where $u$ and $v$ are the in-plane deformation in the $x$ and $y$ directions, respectively, $\lambda$ and $\mu$ are the Lamé elasticity constants, and $\rho$ is the mass per unit volume. The bilinear shape functions result in the classical consistent mass matrix.

$$[M_{el}] = \rho\frac{ab}{36}\begin{bmatrix} 4I & 2I & I & 2I \\ & 4I & 2I & I \\ \text{(symmetric)} & & 4I & 2I \\ & & & 4I \end{bmatrix} \tag{36}$$

The formulation by Stavrinidis of employing the squares of sine and cosine functions to form the mass matrix can be extrapolated to two dimensions:

$$\Phi_1 = \cos^2\alpha \cos^2\beta \tag{37a}$$

$$\Phi_2 = \sin^2\alpha \cos^2\beta \tag{37b}$$

$$\Phi_3 = \sin^2\alpha \sin^2\beta \tag{37c}$$

$$\Phi_4 = \cos^2\alpha \sin^2\beta \tag{37d}$$

where

$$\alpha = (\pi/2)(x/a)$$

$$\beta = (\pi/2)(y/b)$$

The resulting mass matrix is

$$[M] = \rho\frac{ab}{64}\begin{bmatrix} 9I & 3I & I & 3I \\ & 9I & 3I & I \\ & & 9I & 3I \\ \text{(symmetric)} & & & 9I \end{bmatrix} \tag{38}$$

A pseudoaverage matrix is obtained in Eq. (39) by considering that the double integrals appearing in the expression for the mass matrix are separable and that the mass matrix terms are products of the terms in the mass of the one-dimensional bar element:

$$[M] = \rho\frac{ab}{144}\begin{bmatrix} 25I & 5I & I & 5I \\ & 25I & 5I & I \\ & & 25I & 5I \\ \text{(symmetric)} & & & 25I \end{bmatrix} \tag{39}$$

Modal results computed with the pseudomass average in Eq. (39) were found more accurate than those employing the true average of the consistent and lumped-mass matrices:

$$[M] = \rho\frac{ab}{144}\begin{bmatrix} 26I & 4I & I & 4I \\ & 26I & 4I & 2I \\ & & 26I & 4I \\ \text{(symmetric)} & & & 26I \end{bmatrix} \tag{40}$$

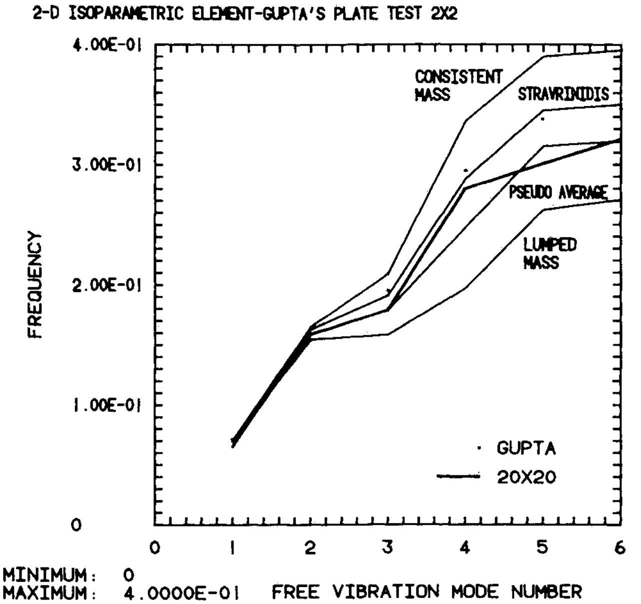


**Fig. 7 Frequency values vs mode number considering various formulations for the two-dimensional model in Fig. 6.**

Since few analytical solutions exist in the literature, the example of Gupta in Ref. 3 was taken. The problem is presented in Fig. 6 and represents a square membrane clamped along one side. A $20 \times 20$ element mesh was chosen as a reference solution for the eigenfrequencies.

The results for the new formulations are presented in Fig. 7. The dynamic element results by Gupta are indicated by an asterisk and are not more precise than the two new formulations in Eqs. (38) and (39). For the first six modes, the maximal frequency error is about 30% for the classical consistent and lumped formulations. This error is reduced to about 10% for the new approach.

Major advantages of the new formulations are that the increased accuracy is acquired at no extra cost and that implementation into existing element libraries is straightforward. A further accuracy improvement might be achieved by allowing $x$-$y$ inertia coupling, and such investigations are in progress.

## Concluding Remarks

Improved finite-element formulations are presented for dynamic analysis. The improvements are based on modifications of the finite-element mass matrix and are of high importance, since no additional computational effort is required, compared with the necessary effort for the standard consistent mass-matrix formulation. For the one-dimensional bar element, a higher-order formulation is derived, yielding considerable accuracy improvement for the frequencies. In general, the frequency errors are at least three times smaller than the corresponding consistent or lumped-mass formulation errors for the first two-thirds of all modes. Furthermore, a formulation has been developed that minimizes the maximum frequency error for all of the problem modes. This approach, referred to as Stavrinidis' formulation, is especially advantageous when a coarse mesh is used, and in general the error is also three times smaller than the corresponding consistent or lumped-mass formluation errors. Free-vibration analysis and forced-vibration results are presented employing the new mass matrices.

Similar to the bar element formulation, a higher-order mass matrix is derived for the pure bending beam element, and a detailed error analysis is presented. The improvements of dynamic results are demonstrated considering a clamped-clamped beam. Similar conclusions are obtained as for the bar element.

Finally, a rectangular two-dimensional membrane element is presented with improved mass matrix and dynamic vibration features. In general, a higher-order formulation is not easily derived for two- and three-dimensional elments. An extension of Stavrinidis' method for the two-dimensional element results in frequencies that are at least as accurate as those computed by the Gupta dynamic elements. Whereas the Gupta method requires the solution of a complex quadratic eigenvalue problem, the present formulation is not more expensive than the consistent mass method. The major advantages of the new formulations are that the increased accuracy is acquired at no extra cost and that implementation into existing element libraries is straightforward.

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